SMOOTHNESS OF DENSITY FOR THE AREA PROCESS OF FRACTIONAL BROWNIAN MOTION

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ABSTRACT. We consider a process given by a two-dimensional fractional Brownian motion with Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$, along with an associated Lévy area, and prove the smoothness of a density for this process with respect to Lebesgue measure.

1. INTRODUCTION

Let $B_t := (B_t^1, B_t^2)$, $t \in [0, T]$ be two-dimensional fractional Brownian motion of Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$, and let

$$(B_m)_t := \left((B_m^1)_t, (B_m^2)_t \right)$$

denote the m-th dyadic approximation of B (as defined below in Section 2.2). Define the area processes

$$(1.1) \qquad (A_m)_t := \frac{1}{2} \left[\int_0^t (B_s^1)_m d(B_s^2)_m - \int_0^t (B_s^2)_m d(B_s^1)_m \right],$$

and $A_t := \lim_{m \to \infty} (A_m)_t$ (where this convergence is almost sure - see Theorem 2 of [7]).

The main result of this paper is as follows:

Theorem 1.1. *Define the random process $\{Y\}_{0 \le t \le T}$, taking values in $\mathbb{R}^3$, by*

$$Y_0 = 0,$$
$$Y_t = (B_t, A_t). \qquad (t \in (0, T])$$

Then for all $t \in [0, T]$, the density of Y_t with respect to Lebesgue measure is C^∞.

The investigation of this process is motivated by the potential for fractional Brownian motion to be a useful driving signal in stochastic differential equations that model a wide variety of natural and financial phenomena; in particular, the presence of long-range persistence (for $H > 1/2$) or anti-persistence (for $H < 1/2$) makes fBm a natural candidate for a driving process in many scenarios. Several examples of such applications are included in [21] and [17].

One area of interest in the study of stochastic differential equations is on finding sufficient conditions for existence and regularity of densities for solutions. More specifically, given some solution $\{Y_t\}$ to the equation

$$(1.2) \qquad\qquad dY_t = \sum_{i=1}^{d} X_i(Y_t) d\xi_t,$$

where $\{X_i\}$ is some collection of vector fields and ξ_t is a Gaussian driving process on the space $\mathcal{C}([0,T], \mathbb{R}^d)$, it is natural ask whether ξ admits a density with respect to Lebesgue measure. It is due to the celebrated theorem of Hörmander (see, for example, Theorem 38.16 of [20]) that, when our driving process is standard Brownian motion, our solution admits a smooth density so long as the set of vectors $\{X_i, [X_i, X_j], [[X_i, X_j], X_k] \ldots\}$ spans $\mathbb{R}^d$.

In the case of fractional Brownian motion, one may no longer appeal to the types of martingale arguments used in proofs of the above result in the standard case. When $H > \frac{1}{2}$, the positive correlation of increments of sample paths results in better variational properties than those of Brownian motion, and so one may use Young's integration theory to attack the problem – existence of a density to a solution of (1.2) under this condition is proven in [19], and smoothness is proven in [1]. When $H < \frac{1}{2}$, one must turn to the rough path theory of T. Lyons (see [14]) in order to interpret (1.2) in a meaningful manner. The connection between fractional Brownian motion with $\frac{1}{4} < H < \frac{1}{2}$ and rough paths is investigated in [7], and existence of a density in the case of $\frac{1}{3} < H\frac{1}{2}$ is proven in [5]. As far as we are aware, Theorem 1.1 is the first positive result involving smoothness, and may give hope that similar results will hold in a more general setting.

In order to prove Theorem 1.1, we appeal to the usual technique of Malliavin calculus. It follows from Theorem 5.1 of [15] that it is enough to show that Y satisfies the following two conditions

1. $Y \in \mathbb{D}^\infty$;
2. If $\gamma = DY(DY)^*$ is the Malliavin covariance matrix associated to Y, then

$$(\det \gamma)^{-1} \in L^{\infty-}(\mathcal{W}^2, \mathbb{P}) := \bigcap_{j \geq 1} L^j(\mathcal{W}^2, \mathbb{P}).$$

We will begin by calculating the derivative DY explicitly, and from this Condition 1 will be proven in Proposition 3.9. Condition 2 will then follow by direct analysis of the Malliavin covariance matrix; see Proposition 3.13 and Corollary 3.15.

2. Background

2.1. **Fractional Brownian Motion.** A (one-dimensional) fractional Brownian motion $\{B_t^H; \ (t \in [0, T])\}$ of Hurst parameter $H \in [0, 1]$ is a continuous-time centered Gaussian process with covariance given by

$$\mathbb{E}(B_s^H B_t^H) = R(s, t) := \frac{1}{2}\left(s^{2H} + t^{2H} - |t - s|^{2H}\right)$$

Our focus will be on fBm with $1/3 < H < 1/2$; henceforth, we shall assume that such an H has been fixed and will drop the parameter from our notation whenever possible to do so without causing confusion.

An n-dimensional fractional Brownian motion is a stochastic process $\{B_t = (B_t^1, \ldots, B_t^n); \ t \in [0, T]\}$ is a continuous-time process comprised of n independent copies of one-dimensional fractional Brownian motion, all having the same Hurst parameter H.

It is straightforward to check that the process B satisfies a self-similarity property; that is to say, the processes B_{at} and $a^{-H}B_t$ are equal in distribution. By Kolmogorov's continuity criterion, the sample paths $t \mapsto B_t$ are almost surely Hölder continuous of order α, for any $\alpha < H$ – see Theorem 1.6.1 of [2] for details of this proof.

2.2. **Dyadic Approximation.** For each m, we will let $D_m := \{k2^{-m}T; k = 0, 1, \ldots, 2^m\}$. We define the m-th dyadic approximator $\pi_m : \mathcal{C}([0, T], \mathbb{R}^d) \longrightarrow \mathcal{C}([0, T], \mathbb{R}^d)$ as the unique projection operator such that, for any given $f \in \mathcal{C}([0, T], \mathbb{R}^d)$,

$$\pi_m f(t) = f(t), \quad (t \in D_m)$$
$$\frac{d^2}{dt^2}\pi_m f(t) = 0. \quad (t \notin D_m)$$

In words, $\pi_m f$ is nothing more than the piecewise linear path agreeing with f on the set D_m. We will regularly use the shorthand notation $f_m := \pi_m f$ where convenient. Similarly, we will define the m-th dyadic approximation of fractional Brownian motion $B_m := \pi_m B$; more explicitly,

$$(B_m)_t := B_{t_-} + (t - t_-)2^m[B_{t_+} - B_{t_-}], \quad (0 \le t \le T)$$

where t_- is the largest member of D_m such that $t_- \le t$ and t_+ is the smallet member of D_m such that $t \le t_+$.

2.3. p**-variation and Rough Paths.** Let U be a Banach space (U will typically be $\mathbb{R}$ or $\mathbb{R}^2$) with norm $\| \cdot \|_U$, and $\mathcal{P}[0, T]$ denote the set of finite partitions of $[0, T]$. Suppose we are given a path $f \in \mathcal{C}([0, T], U)$;

then for each $1 \leq p < \infty$ and $\Pi = \{0 = t_1, t_2, \ldots, t_N = 1\} \in \mathcal{P}[0, T]$, one may define the quantities

$$\Delta_i f := f(t_i) - f(t_{i-1}),$$

$$V_p(f : \Pi) := \left(\sum_{i=1}^{N} \|\Delta_i f\|_U^p \right)^{\frac{1}{p}},$$

$$\|f\|_p := \sup_{\Pi \in \mathcal{P}[0,T]} V_p(f : \Pi).$$

The norm $\| \cdot \|_p$ is referred to as the *p-variation norm*; we shall define the space $\mathcal{C}_p(U) := \{f \in \mathcal{C}([0, T], U); \|f\|_p < \infty\}$, which is a Banach space under $\| \cdot \|_p$. It is easy to check that for given α and p such that $\alpha < \frac{1}{p}$, any α-Hölder continuous function is in $\mathcal{C}_p(U)$. Also, one has that for any $1 \leq p < q$, $\mathcal{C}_p(U) \subset \mathcal{C}_q(U)$.

Given $f \in \mathcal{C}_p(U), g \in \mathcal{C}_q(U)$, where p and q are such that $\frac{1}{p} + \frac{1}{q} > 1$, one can develop the notion of integration of f against g in the following manner: if $\{\Pi_n := \{t_i\}\} \subset \mathcal{P}[0, T]$ is a collection of partitions such that the mesh size $|\Pi_n|$ tends to 0 as $n \to \infty$, we define

$$\int_0^T f \, dg := \lim_{n \to \infty} \sum_{i=1}^{\#(\Pi_n)} f(c_i) \Delta_i g$$

where $c_i \in (t_{i-1}, t_i)$. This limit is guaranteed to exist under the assumptions presented, and is independent of the choice we make of the family of partitions so long as their mesh size tends to zero - see Theorem 3.3.1 of [13] for further details. The element $\int_0^1 f \, dg$ is referred to as *the Young's integral of f against g*. This expression was originally formulated in [26]. We have the following estimate on the value of this expression (see Formula 10.9 of [26])

$$(2.1) \qquad \left| \int_0^T f \, dg - [f(0) \cdot (g(T) - g(0))] \right| \leq C\|f\|_p\|g\|_q,$$

where the constant C depends only on the values of p and q.

Similarly, given some $f \in \mathcal{C}([0, T]^2, U)$, and partitions $\Pi_1 = \{s_i\}$, $\Pi_2 = \{t_j\} \in \mathcal{P}[0, T]$, we may define

$$\Delta_{ij} f := f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1}),$$

$$V_p(f : \Pi_1, \Pi_2) := \left(\sum_{i=1}^{\#(\Pi_1)} \sum_{j=1}^{\#(\Pi_2)} \|\Delta_{ij} f\|_U^p \right)^{\frac{1}{p}},$$

$$\|f\|_p^{(2D)} := \sup_{\Pi_1, \Pi_2 \in \mathcal{P}[0,T]} V_p(f : \Pi_1, \Pi_2).$$

As in the (one-dimensional) case above, we shall define the space $\mathcal{C}_p^{(2D)}(U) := \{f \in \mathcal{C}([0,T]^2, U); \ \|f\|_p^{(2D)} < \infty\}$. It is helpful to note that, for each $f \in \mathcal{C}_p^{(2D)}(U)$ such that $f(0, \cdot) = 0$, then for each fixed $s \in [0,T]$, $f(s, \cdot) \in \mathcal{C}_p(U)$ and $\|f(s, \cdot)\|_p \le \|f\|_p^{(2D)}$, since

$$\sum_{i=1}^{\#(\Pi)} |\Delta_i f(s, \cdot)|^p = \sum_{i=1}^{\#(\Pi)} |\Delta_i f(s, \cdot) - \Delta_i f(0, \cdot)|^p$$

$$\le \sum_{i=1}^{\#(\Pi)} \big(|\Delta_i f(s, \cdot) - \Delta_i f(0, \cdot)|^p$$

$$+ |\Delta_i f(T, \cdot) - \Delta_i f(s, \cdot)|^p\big) \le \big(\|f\|_p^{(2D)}\big)^p.$$

Trivially, one also has that for each $f \in \mathcal{C}_p(U)$, the function $f \otimes f : [0,T]^2 \to U$ defined by

$$(f \otimes f)(s,t) := f(s)f(t)$$

is contained in $\mathcal{C}_p^{(2D)}(U)$, and $\|f \otimes f\|_p^{(2D)} \le \|f\|_p^2$.

Given $f \in \mathcal{C}_p^{(2D)}(U)$ and $g \in \mathcal{C}_q^{(2D)}(U)$, where p and q are such that $\frac{1}{p}+\frac{1}{q} > 1$, the _2D-Young's integral of f against g_, denoted by $\int_{[0,T]^2} f \ dg$, is defined by

$$\int_{[0,T]^2} f \ dg := \lim_{n\to\infty} \sum_{i=1}^{\#(\Pi_n)} \sum_{j=1}^{\#(\Psi_n)} f(c_i, d_j)\Delta_{ij}g,$$

where, as before, $\{\Pi_n := \{t_i\}\}, \{\Psi_n := \{s_j\}\} \subset \mathcal{P}[0,T]$ are collections of partitions such that the maximum mesh size $|\Pi_n| \vee |\Psi_n|$ tends to 0 as $n \to \infty$ and $c_i \in (t_{i-1}, t_i)$, $d_j \in (s_{j-1}, s_j)$. Existence of this limit under the given assumptions, independent of the choice of the family of partitions, is proven in Theorem 1.2 of [23], as is an estimate similar to that of the one-dimensional case:

$$\left| \int_{[0,T]^2} f \ dg \right| \le C\|g\|_q^{(2D)} \Big(\|f\|_p^{(2D)} + \|f(0, \cdot)\|_p + \|f(\cdot, 0)\|_p + |f(0,0)| \Big).$$

It will be helpful to record here a pair of results relating the variation of paths with their linear approximations.

Theorem 2.1 (Propositions 5.20 and 5.60 of [11])**.**

    (1) _Suppose $x \in \mathcal{C}_p(U)$, and let $x_m := \pi_m x$ be the dyadic approximation to x as defined above. Then one has that_

$$\|x_m\|_p \le 3^{p-1}\|x\|_p.$$

(2) *Suppose $x \in \mathcal{C}_p^{(2D)}(U)$, and let $x_m := \pi_m x$ be the dyadic approximation to x as defined above. Then one has that*

$$\|x_m\|_p^{(2D)} \leq 9^{p-1}\|x\|_p^{(2D)}.$$

For further development of the theory, the interested reader may look in [26], [9], and [23].

2.4. **Gaussian measure spaces.** Let $(\mathcal{W}, \|\cdot\|)$ denote a separable Banach space. We will say that a measure $\mathbb{P}$ on $\mathcal{W}$ is Gaussian if there exists a symmetric bilinear form $q : \mathcal{W}^* \times \mathcal{W}^* \longrightarrow \mathbb{R}$ such that for all $\varphi \in \mathcal{W}^*$,

$$\int_{\mathcal{W}} \exp\left(i\varphi(\omega)\right) \, d\mathbb{P}(\omega) = \exp\left(-\frac{1}{2}q(\varphi, \varphi)\right).$$

Let $\mathcal{B}$ refer to the Borel σ-algebra on $\mathcal{W}$; we will call the triple $(\mathcal{W}, \mathcal{B}, \mathbb{P})$ a Gaussian space. Define a continuous mapping $J : L^2(\mathbb{P}) \to \mathcal{W}$ by

$$Jf := \int_{\mathcal{W}} \omega f(\omega) \, d\mathbb{P}(\omega).$$

Define $\mathcal{H}$ as the image of J restricted to the space $\overline{\mathcal{W}^*}^{L^2(\mathbb{P})}$; this space may be equipped with inner product given by

$$\langle Jf, Jg \rangle_{\mathcal{H}} = \langle f, g \rangle_{L^2(\mathbb{P})}.$$

We will refer to $\mathcal{H}$ as the *Cameron-Martin space* associated to the Gaussian space $(\mathcal{W}, \mathcal{B}, \mathbb{P})$. More information regarding the construction of these spaces may be found in [3], [8], and [12].

A well-known example of a Gaussian measure space is the one associated to Brownian motion where

$$\mathcal{W} := \{\omega \in \mathcal{C}([0, T], \mathbb{R}) : \omega(0) = 0\}$$

and the Gaussian measure $\mathbb{P}$ on $\mathcal{W}$ is the law of standard Brownian motion. Details of the construction of this measure may be found in [24] and [25].

In this case, the Cameron-Martin space is

$$\mathcal{H} = \mathcal{H}_{\frac{1}{2}} := \left\{ h \in \mathcal{W} : h(s) = \int_0^s \phi(u) \, du; \phi \in L^2[0, T] \right\}$$

with inner product given by

$$\langle h, k \rangle_{\mathcal{H}_{\frac{1}{2}}} := \int_0^t h'(s)k'(s)ds.$$

A second example of a Gaussian measure space which is pertinent to the results described below, is as follows: let $\mathcal{W}$ be defined as above, and define the Gaussian measure $\mathbb{P}$ on $\mathcal{W}$ as the law of fractional Brownian

motion with Hurst parameter $H < 1/2$; then by following Proposition 2.1.2 of [2], we have that $\mathcal{H}$ consists of functions of the form $h(t) = \int_0^t K_H(t,s)\hat{h}(s)\,ds$, where $\hat{h} \in L^2[0,T]$ and

$$
K_H(t,s) := b_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} \right.
$$
$$
\left. - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)H - \frac{1}{2}u^{H-\frac{3}{2}}\,du \right],
$$

where b_H is some suitable normalization constant. The inner product of this space is given by

$$
\langle h, k \rangle_{\mathcal{H}} := \langle \hat{h}, \hat{k} \rangle_{L^2[0,T]}.
$$

For each fixed $t \in [0,T]$, the function $R(t,\cdot) = \mathbb{E}[B_t B_\cdot] \in \mathcal{H}$ is a *reproducing kernel* for the space; that is to say, for any $h \in \mathcal{H}$, we have the following:

$$
\langle h, R(t,\cdot) \rangle_{\mathcal{H}} = h(t).
$$

Let $\mathcal{S}$ refer to the space of *cylinder functions*; that is to say, random variables of the form

$$
F(\omega) = f(\phi_1(\omega), \dots, \phi_n(\omega)),
$$

where $f \in C^\infty(\mathbb{R}^n)$ with all partial derivatives having at most polynomial growth, and $\{\phi_1, \dots, \phi_n\} \subset \mathcal{W}^*$. Let $D : \mathcal{S} \to \mathcal{S} \otimes \mathcal{H}^*$ be the operator defined by the action by

$$
DF(\omega)k := \frac{d}{dt}|_{t=0} F(\omega + tk)
$$
$$
= \sum_{i=1}^n \partial_i f(\phi_1(\omega), \dots, \phi_n(\omega))\phi_i(k) \quad (k \in \mathcal{H}).
$$

For $1 \le q < \infty$, we will let $\mathbb{D}^{1,q}$ denote the closure of $\mathcal{S}$ with respect to the norm

$$
\|F\|_{1,q} := (\mathbb{E}[|F|^q] + \mathbb{E}[\|DF\|_{\mathcal{H}^*}^q])
$$

One can naturally define an iterated derivate operator D^k taking values in $\mathcal{H}^{\otimes k}$; from this we can define the seminorm

$$
\|F\|_{k,q} := \left(\mathbb{E}[|F|^q] + \sum_{j=1}^k \mathbb{E}[\|D^j F\|_{\mathcal{H}^{\otimes k}}^q] \right)
$$

and we will denote by $\mathbb{D}^{k,q}$ the closure of $\mathcal{S}$ with respect to $\|\cdot\|_{k,q}$. Also, let

$$\mathbb{D}^\infty := \bigcap_{k\in\mathbb{N}}\bigcap_{q\geq 1}\mathbb{D}^{k,q}.$$

Given some $F \in \mathbb{D}^{1,q}$, we may define the *Malliavin covariance matrix* γ by

$$\gamma := DF(DF)^*.$$

3. Proof of Theorem 1.1

Again, we fix the Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$. Let

$$\mathcal{W}^2 := \mathcal{W} \otimes \mathbb{R}^2 \cong \mathcal{W} \times \mathcal{W} = \{\omega \in \mathcal{C}([0,T],\mathbb{R}^2) : \omega(0) = \mathbf{0}\}.$$

On $\mathcal{W}^2$, one may construct a unique Gaussian measure $\mathbb{P}$ such that the coordinate process $\{B_t\}_{0\leq t\leq T}$ defined by

$$B_t(\omega) = \omega(t)$$

is a two-dimensional fractional Brownian motion with Hurst parameter H and $\mathbb{P} = \text{Law}(B)$. Our reproducing kernel Hilbert space in this instance is given as $\mathcal{H}^2 = \mathcal{H} \otimes \mathbb{R}^2$, where $\mathcal{H}$ is the Cameron-Martin space for one-dimensional fractional Brownian motion; for a general element $h = (h^1, h^2) \in \mathcal{H}^2$, the norm is given by $\|h\|_{\mathcal{H}^2}^2 = \|h^1\|_{\mathcal{H}}^2 + \|h^2\|_{\mathcal{H}}^2$.

We shall fix

$$(3.1) \qquad\qquad p \in \left(\frac{1}{H}, \frac{1}{1-2H}\right),$$

and define (following Section 5.3.3 of [11]) the spaces

$$\mathcal{W}_p := \overline{C^\infty([0,T],\mathbb{R}) \cap \mathcal{W}}^{\|\cdot\|_p};$$

$$\mathcal{W}_p^2 := \mathcal{W}_p \times \mathcal{W}_p \cong \overline{C^\infty([0,T],\mathbb{R}^2) \cap \mathcal{W}^2}^{\|\cdot\|_p}.$$

For reasons which will become apparent as we progress, it will be beneficial for us to declare that from here on out our process $\{B_t\}$ will be restricted to the probability space $(\mathcal{W}_p^2, \mathcal{B}_{\mathcal{W}_p^2}, \mathbb{P}|_{\mathcal{W}_p^2})$; the details of this restriction are included in the Appendix. Most importantly, the Cameron-Martin space $\mathcal{H}^2$ associated to the restriction of our measure is the same as the Cameron-Martin space associated to $(\mathcal{W}^2, \mathcal{B}, \mathbb{P})$ as given previously. The following proposition shows that elements of $\mathcal{H}$ live within a smaller variational space, and will be used repeatedly in the sequel.

Proposition 3.1. *Let $r := \frac{1}{2H}$. Then the covariance kernel R associated to fractional Brownian motion with Hurst parameter H has finite two-dimensional r-variation. Also, the associated reproducing kernel*

Hilbert space $\mathcal{H}$ may be embedded in the space $\mathcal{C}_r(\mathbb{R})$; furthermore, this embedding is a contraction.

Remark 3.2. *The assertions in Proposition 3.1 were originally made in p. 3363 of [5] or p.13 of [10]. However, some technical gaps have been found in their proof that $R \in \mathcal{C}_r^{(2D)}$; as a result, we have included a direct proof of this statement within the Appendix (see Theorem 4.3 below).*

Note that the above implies that $\mathcal{H}^2 \subset \mathcal{C}_{\tilde{p}}(\mathbb{R}^2)$, and, since $r < p$, $\mathcal{H}^2 \subset \mathcal{W}_p^2$. Also note that Proposition 3.1 implies that $R(s, \cdot) \in \mathcal{C}_r$ for each fixed $s \in [0, T]$.

Let Y_m be the solution to the differential equation

$$d(Y_m)_s = X_1(Y_m)_s d(B_m^1)_s + X_2(Y_m)_s d\left(B_m^2\right)_s$$
$$(Y_m)_0 = \mathbf{0}$$

with vector fields given by $X_1 := \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}$, $X_2 := \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}$. Without a loss of generality, we may fix all of the processes in question at time T, and will elect to do so for the remainder. Explicit calculations show that

$$(Y_m)_T = \left((B_m)_T, \frac{1}{2}\int_0^T (B_m^1)_s \, d(B_m^2)_s - (B_m^2)_s \, d(B_m^1)_s\right).$$

As a result of Theorem 19 and Corollary 20 of [7], we have the existence of a process

$$Y_T := \lim_{m\to\infty} (Y_m)_T; \quad (\text{a.s. and in } L^2)$$

we may denote this process suggestively as

$$Y_T = \left(B_T, \frac{1}{2}\int_0^T \left(B_s^1 \, dB_s^2 - B_s^2 \, dB_s^1\right)\right) = \left(B_T, \frac{1}{2}\int_0^T \alpha\left(B_s, dB_s\right)\right)$$

(In this expression we have let $\alpha\left(a, b\right) := a^1 b^2 - a^2 b^1$.)

3.1. Calculation of the Derivative of Y_T.

So that we might study the properties of the processes $(Y_m)_T$ and Y_T, let us introduce the following expressions: given some $\omega := \omega^1 e_1 + \omega^2 e_2 \in \mathcal{W}_p^2$, we will let $\tilde{\omega}$ be the element of $\mathcal{W}^2$ defined by

$$\tilde{\omega} := \omega^2 e_1 - \omega^1 e_2 = J\omega,$$

where J is rotation by $-\pi/2$. We will let $q : \mathcal{H}^2 \times \mathcal{H}^2 \to \mathbb{R}$ denote the symmetric quadratic form given by

$$(3.2) \qquad q(h, k) = \frac{1}{2} \left(\int_0^T h(t) \cdot d\tilde{k}(t) + \int_0^T k(t) \cdot d\tilde{h}(t) \right)$$

$$= \int_0^T h(t) \cdot d\tilde{k}(t) + \frac{1}{2} \left(k(T) \cdot \tilde{h}(T) \right).$$

The integrals given in the definition of q are to be considered as Young's integrals; by our previous assumption that $\frac{1}{3} < H < \frac{1}{2}$, we have that $\frac{2}{r} = 4H > 1$, and so q is well-defined. Furthermore, standard Young's integration bounds and Proposition 3.1 gives us that

$$(3.3) \qquad |q(h, k)| \leq C\|h\|_r\|k\|_r \leq C\|h\|_{\mathcal{H}}\|k\|_{\mathcal{H}},$$

and hence q is continuous in each variable. We may now rewrite our approximate solutions Y_m in the following form:

$$Y_m = \left((B_m)_T, \frac{1}{2}q(B_m, B_m) \right).$$

As a consequence of the representation theorem of Riesz, we have the existence of a linear operator $Q : \mathcal{H}^2 \to \mathcal{H}^2$ such that $\langle Qh, k \rangle = q(h, k)$.

Remark 3.3. *Recall that $r = \frac{1}{2H}$; thus, our assumption on the value of p as made in (3.1) is sufficient to guarantee that the integrals above are well-defined, since this implies that $\frac{1}{p} + \frac{1}{r} > 1$.*

Proposition 3.4. *Fix $\alpha \in \mathcal{W}_p$; for each partition $\Pi = \{t_i\}_{i=0}^N \in \mathcal{P}[0, T]$, define the vector $S_\Pi \in \mathcal{H}$ in the following manner:*

$$S_\Pi(\cdot) := \sum_{i=1}^N \alpha(c_i) \left[R(t_i, \cdot) - R(t_{i-1}, \cdot) \right],$$

where $c_i \in (t_{i-1}, t_i)$. Then $\mathcal{H} - \lim_{k \to \infty} S_{\Pi_k}$ exists, where $\{\Pi_k\}_{k=1}^\infty \subset \mathcal{P}[0, T]$ with $|\Pi_k|$ converging to zero as $k \longrightarrow \infty$; furthermore, this limit is independent of the family of partitions. We will denote this limit by

$$\int_0^T \alpha(t) R(dt, \cdot).$$

This limit satisfies the following properties:

(1) $\left\| \int_0^T \alpha(t) R(dt, \cdot) \right\|_{\mathcal{H}}^2 = \int_{[0,T]^2} \alpha \otimes \alpha \, dR$; hence, there exists a constant $C > 0$ such that

$$\left\| \int_0^T \alpha(t) R(dt, \cdot) \right\|_{\mathcal{H}}^2 \le C \|\alpha\|_p^2 \|R\|_r^{(2D)}.$$

(2) For each $h \in \mathcal{H}$, $\left\langle \int_0^T \alpha(t) R(dt, \cdot), h \right\rangle_{\mathcal{H}} = \int_0^T \alpha(t) dh(t)$.

(3) $\left(\int_0^T \alpha(t) R(dt, \cdot) \right)(s) = \int_0^T \alpha(t) R(dt, s)$.

Proof. First note that $\frac{1}{p} + \frac{1}{r} > (1 - 2H) + 2H = 1$, which implies that

   (1) the Young's integral of α against $R(\cdot, s)$ for any $s \in [0, T]$ is well-defined, and

   (2) the 2D-Young's integral of $\alpha \otimes \alpha$ against R is well-defined.

It follows from Theorem 2.1 of [23] that for each k,

$$(3.4) \qquad \|S_{\Pi_k}\|_{\mathcal{H}}^2 = \sum_{i=1}^{\#(\Pi_k)} \sum_{j=1}^{\#(\Pi_k)} \alpha(c_i) \alpha(c_j) \langle \Delta_i R(t_i, \cdot), \Delta_j R(t_j, \cdot) \rangle_{\mathcal{H}}$$

$$\le C \|\alpha\|_p^2 \|R\|_r^{(2D)},$$

where C is a constant depending only on p and r. Given any two partitions $\Pi_n = \{s_i\}, \Pi_m = \{t_k\}$ in the family, for $c_i \in [s_{i-1}, s_i], d_k \in [t_{k-1}, t_k]$,

$$\|S_{\Pi_n} - S_{\Pi_m}\|_{\mathcal{H}}^2 = \|S_{\Pi_n}\|_{\mathcal{H}}^2 + \|S_{\Pi_m}\|_{\mathcal{H}}^2 - 2\langle S_{\Pi_n}, S_{\Pi_m} \rangle_{\mathcal{H}}$$

$$= \sum_{i=1}^{\#(\Pi_n)} \sum_{j=1}^{\#(\Pi_n)} \alpha(c_i) \alpha(c_j) [\Delta_{ij} R(s_i, s_j)]$$

$$+ \sum_{k=1}^{\#(\Pi_m)} \sum_{l=1}^{\#(\Pi_m)} \alpha(d_k) \alpha(d_l) [\Delta_{kl} R(t_k, t_l)]$$

$$- 2 \sum_{i=1}^{\#(\Pi_n)} \sum_{k=1}^{\#(\Pi_m)} \alpha(c_i) \alpha(d_k) [\Delta_{ik} R(s_i, t_k)]$$

$$\xrightarrow{n,m \to \infty} \int_{[0,T]^2} (\alpha \otimes \alpha)(s, t) \, dR(s, t)$$

$$+ \int_{[0,T]^2} (\alpha \otimes \alpha)(s, t) \, dR(s, t)$$

$$- 2 \int_{[0,T]^2} (\alpha \otimes \alpha)(s, t) \, dR(s, t) = 0.$$

Hence, the completeness of $\mathcal{H}$ implies the existence of $\int_0^T \alpha(t)R(dt,\cdot)$. Since the 2D-Young's integral is independant of choice of partitions, one may also see from the calculation above that the limit of S_{Π_k} is also independant of choice of partition, as claimed. Letting k tend to infinity in (3.4) and applying bounds as in Theorem 1.2 of [23] proves (1). For an arbitrary $h \in \mathcal{H}$, we note that

$$
\left\langle \int_0^T \alpha(t)\, R(dt,\cdot), h \right\rangle_{\mathcal{H}} = \lim_{|\Pi|\to 0} \langle S_\Pi, h \rangle_{\mathcal{H}}
$$

$$
= \lim_{|\Pi|\to 0} \sum_{i=1}^{\#(\Pi)} \alpha(c_i)\big\langle R(t_{i+1},\cdot) - R(t_i,\cdot), h \big\rangle_{\mathcal{H}}
$$

$$
= \lim_{|\Pi|\to 0} \sum_{i=1}^{\#(\Pi)} \alpha(c_i)\left[h(t_{i+1}) - h(t_i) \right]
$$

$$
= \int_0^T \alpha(t)\, dh(t),
$$

and so (2) holds. In particular, by setting $h = R(s,\cdot)$, (3) is a consequence of (2). $\qquad \square$

Proposition 3.5. *Let* $Q : \mathcal{H}^2 \to \mathcal{H}^2$ *be the bounded operator defined by*

$$
q(h,k) = \langle Qh, k \rangle_{\mathcal{H}^2}.
$$

Then the action of Q *on elements of* $\mathcal{H}^2$ *is given by*

$$
(3.5) \qquad Qh := \frac{1}{2} R\left(T,\cdot\right) \tilde{h}\left(T\right) - \int_0^T \tilde{h}\left(t\right) R\left(dt,\cdot\right).
$$

Proof. Pick an arbitrary $k \in \mathcal{H}^2$. The inner product of k against each of the terms on the right hand side of (3.5) is given as

$$
\left\langle R\left(T,\cdot\right)\tilde{h}\left(T\right), k \right\rangle_{\mathcal{H}^2} = \tilde{h}^1(T)\left\langle R\left(T,\cdot\right), k^1 \right\rangle_{\mathcal{H}} + \tilde{h}^2(T)\left\langle R\left(T,\cdot\right), k^2 \right\rangle_{\mathcal{H}}
$$

$$
= \tilde{h}^1(T)k^1(T) + \tilde{h}^2(T)k^2(T) = \tilde{h}(T)\cdot k(T),
$$

and, as a result of Proposition 3.4,

$$
\left\langle \int_0^T \tilde{h}(t) R(dt, \cdot), k \right\rangle_{\mathcal{H}^2} = \left\langle \int_0^T \tilde{h}^1(t) R(dt, \cdot), k^1 \right\rangle_{\mathcal{H}}
$$
$$
+ \left\langle \int_0^T \tilde{h}^2(t) R(dt, \cdot), k^2 \right\rangle_{\mathcal{H}}
$$
$$
= \int_0^T \tilde{h}^1(t) \, dk^1(t) + \int_0^T \tilde{h}^2(t) \, dk^2(t)
$$
$$
= \int_0^T \tilde{h}(t) \cdot dk(t) = - \int_0^T h(t) \cdot d\tilde{k}(t).
$$

By combining these terms and comparing to (3.2), we see that the claim is proven. $\qquad\square$

Proposition 3.6. *Let $Q : \mathcal{H}^2 \longrightarrow \mathcal{H}^2$ be the operator defined above.*

1. *Q may be extended to an operator from $\mathcal{W}_p^2$ into $\mathcal{H}^2$, which will also be denoted by Q; for any $\omega \in \mathcal{W}_p^2$,*

$$
Q\omega := \frac{1}{2} R(T, \cdot)\tilde{\omega}(T) - \int_0^T \tilde{\omega}(t) \, R(dt, \cdot).
$$

2. *Q is a bounded operator on $\mathcal{W}_p^2$.*

Proof. 1. That Q is well-defined as an operator on $\mathcal{W}_p^2$ follows from Proposition 3.4; it is then of immediate consequence that $Q\omega \in \mathcal{H}^2$ for any $\omega = (\omega^1, \omega^2) \in \mathcal{W}_p^2$.

2. From this fact, along with the estimate given by (3.4), the value of $\|Q\omega\|_{\mathcal{H}^2}^2$ for a fixed $\omega \in \mathcal{W}_p^2$ may be calculated:

$$\|Q\omega\|_{\mathcal{H}^2}^2 = \sum_{i=1}^{2} \left\| \frac{1}{2}R(T,\cdot)\tilde{\omega}^i(T) - \int_0^T \tilde{\omega}^i(t)\,R(dt,\cdot) \right\|_{\mathcal{H}}^2$$

$$= \sum_{i=1}^{2} \frac{1}{4}\left\| R(T,\cdot)\tilde{\omega}^i(T) \right\|_{\mathcal{H}}^2 + \left\| \int_0^T \tilde{\omega}^i(t)\,R(dt,\cdot) \right\|_{\mathcal{H}}^2$$

$$\qquad - \left\langle R(T,\cdot)\tilde{\omega}^i(T), \int_0^T \tilde{\omega}^i(T)\,R(dt,\cdot) \right\rangle_{\mathcal{H}}$$

$$= \sum_{i=1}^{2} \frac{T^{2H}|\tilde{\omega}^i(T)|^2}{4} + \left\| \int_0^T \tilde{\omega}^i(t)\,R(dt,\cdot) \right\|_{\mathcal{H}}^2$$

$$\qquad - \tilde{\omega}^i(T)\int_0^T \tilde{\omega}^i(t)\,R(dt,T)$$

$$\leq \sum_{i=1}^{2} \frac{T^{2H}|\tilde{\omega}^i(T)|^2}{4} + \left| \int_{[0,T]^2} \left(\tilde{\omega}^i \otimes \tilde{\omega}^i\right)(s,t)\,dR(s,t) \right|$$

$$\qquad + \left| \int_0^T \tilde{\omega}^i(T)\tilde{\omega}^i(t)\,R(dt,T) \right|$$

$$\leq \sum_{i=1}^{2} \|\tilde{\omega}^i\|_p^2 \left(\frac{T^{2H}}{4} + \|R\|_r^{(2D)} + \|R(T,\cdot)\|_r \right)$$

$$\leq \left(\frac{T^{2H}}{4} + 2\|R\|_r^{(2D)} \right)\|\omega\|_p^2,$$

which is finite by Proposition 3.1.

$\square$

Let us denote by QB the random variable taking values in $\mathcal{H}^2$:

$$QB := \frac{1}{2}R(T,\cdot)\tilde{B}_T - \int_0^T \tilde{B}_t\,R(dt,\cdot),$$

where $\tilde{B} := (B^2, -B^1)$.

We are now in a position to calculate the derivative of the process Y_T. To begin with, for $i = 1, 2$ let us denote by R_t^i the linear operator on $\mathcal{H}^2$ with action given by

$$R_t^i h = \langle R(t,\cdot)e_i, h \rangle e_i = h^i(t)e_i,$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. It is immediate that $DB = R_T^1 + R_T^2$.

Proposition 3.7. *The process Y_T has a derivative, DY_T, taking values in the space of linear operators from $\mathcal{H}^2$ into $\mathbb{R}^3$, with action given by*

$$DY_T h = \left(R_T^1 h + R_T^2 h, \langle QB, h \rangle_{\mathcal{H}^2} \right) \quad a.s.$$

Proof. Recall that $Y_T = (B_T, A_T)$, where A_T was defined as the almost sure limit of processes given by (1.1). We have that Y_T is continuously $\mathcal{H}^2$-differentiable by [6, Proposition 3], and Corollaries 16 and 20 of [7] imply that $\mathbb{E}\,|(Y_m)_T - Y_T|^2 \to 0$ as $m \to \infty$. We claim that $DA_T h = \langle QB, h \rangle$; to prove this, it suffices to show that

$$(3.6) \qquad \mathbb{E}\left\| \langle QB, \cdot \rangle - D\left(\frac{1}{2} q(B_m, B_m) \right) \right\|^2_{(\mathcal{H}^2)^*} \overset{m \to \infty}{\longrightarrow} 0.$$

Recall that the process B_m was defined as the dyadic linear approximator to our fractional Brownian motion B; similarly, we will denote by $R_m(u, v)$ the m-th dyadic approximation of the kernel $R \in \mathcal{H}$ in the first variable; i.e.,

$$R_m(u, v) := \pi_m(R(\cdot, v))(u).$$

Since $B(T) \cdot \tilde{B}(T) = 0$, it follows from (3.2) that

$$q(B_m, B_m) = \int_0^T (B_m)_t \cdot d(\tilde{B}_m)_t$$

$$= \frac{1}{2} \sum_{k=0}^{2^m} \left(B_{t_i} + B_{t_{i-1}} \right) \cdot \left(\tilde{B}_{t_i} - \tilde{B}_{t_{i-1}} \right).$$

By the product rule (see Proposition 1.2.3 of [18], for example),

$$Dq(B_m, B_m)h = \frac{1}{2} \sum_{k=0}^{2^m} \left[\left(h(t_i) + h(t_{i-1}) \right) \cdot \left(\tilde{B}_{t_i} - \tilde{B}_{t_{i-1}} \right) \right.$$

$$\left. + \left(B_{t_i} + B_{t_{i-1}} \right) \cdot \left(\tilde{h}(t_i) - \tilde{h}(t_{i-1}) \right) \right]$$

$$= \int_0^T h_m(t) \cdot d(\tilde{B}_m)_t + \int_0^T (B_m)_t \cdot d\tilde{h}_m(t)$$

$$= h(T) \cdot \tilde{B}_T - 2 \int_0^T (\tilde{B}_m)_t \cdot dh_m(t).$$

$$= \left\langle R(T, \cdot)\tilde{B}_T - 2 \int_0^T (\tilde{B}_m)_t \, R_m(dt, \cdot), h \right\rangle_{\mathcal{H}^2}.$$

Since $\|\langle h, \cdot \rangle\|_{\mathcal{H}^*} = \|h\|_{\mathcal{H}}$ for any Hilbert space $\mathcal{H}$, we can use the above calculations to rewrite the left side of (3.6) as

$$\mathbb{E} \left\| QB - \left(\frac{1}{2}R(T, \cdot)\tilde{B}_T - \int_0^T (\tilde{B}_m)_t \, R_m(dt, \cdot) \right) \right\|_{\mathcal{H}^2}^2$$

$$= \mathbb{E} \left\| \int_0^T \tilde{B}_t \, R(dt, \cdot) - \int_0^T (\tilde{B}_m)_t \, R_m(dt, \cdot) \right\|_{\mathcal{H}^2}^2.$$

Hence, to prove the claim is it required for us to show that

$$(3.7) \qquad \mathbb{E} \left\| \int_0^T B_t^i R(dt, \cdot) - \int_0^T (B_m^i)_t R_m(dt, \cdot) \right\|_{\mathcal{H}}^2 \stackrel{m \to \infty}{\longrightarrow} 0.$$

We note that for any $h \in \mathcal{H}$,

$$\lim_{m \to \infty} \left\langle \int_0^T (B_m)_t \, R_m(dt, \cdot), h \right\rangle_{\mathcal{H}}$$

$$= \frac{1}{2} \left[\lim_{m \to \infty} \sum_{k=0}^{2^m} \left((B_m)_{t_k} + (B_m)_{t_{k-1}} \right) \left(h(t_k) - h(t_{k-1}) \right) \right]$$

$$= \frac{1}{2} \left[\lim_{m \to \infty} \sum_{k=0}^{2^m} (B_m)_{t_k} \left(h(t_k) - h(t_{k-1}) \right) \right.$$

$$\left. + (B_m)_{t_{k-1}} \left(h(t_k) - h(t_{k-1}) \right) \right]$$

$$= \frac{1}{2} \left[2 \int_0^T B_t \, dh(t) \right] = \left\langle \int_0^T B_t \, R(dt, \cdot), h \right\rangle_{\mathcal{H}}$$

and so $\int_0^T (B_m)_t \, R_m(dt, \cdot)$ converges pointwise in $\mathcal{H}$ to $\int_0^T B_t \, R(dt, \cdot)$.

Applying Proposition 3.4, along with Theorem 2.1, we may conclude that there exists a constant C for which

$$\left\| \int_0^T B_t \, R(dt, \cdot) \right\|_{\mathcal{H}}^2 \leq C \|B\|_p^2 \|R\|_r^{(2D)},$$

$$\left\| \int_0^T (B_m)_t \, R_m(dt, \cdot) \right\|_{\mathcal{H}}^2 \leq C \|B\|_p^2 \|R\|_r^{(2D)},$$

with the second inequality being independent of m. Hence, we may use Fernique's Theorem (Theorem 2.6 of [8]) to conclude that

$$(3.8) \qquad \left\| \int_0^T B_t \, R(dt, \cdot) - \int_0^T (B_m)_t \, R_m(dt, \cdot) \right\|_{\mathcal{H}}^2 \leq C \|B\|_p^2 < \infty.$$

Hence, we may apply Dominated Convergence to conclude that (3.7) holds, as desired.

$\square$

Remark 3.8. *In fact, we have shown something slightly stronger in the above proof. By changing the exponent on the left-hand side of (3.8), we may conclude that $\|D(Y_m)_T - DY_T\|_{(\mathcal{H}^2)^*}$, converges to zero in all $L^j, j \geq 1$. By applying the triangle inequality, we also find that $\|DY_T\|_{(\mathcal{H}^2)^*} \in L^{\infty-}$.*

Proposition 3.9. *The random variable Y_T is in $\mathbb{D}^\infty$.*

Proof. Corollaries 16 and 20 of [7] implies that $\mathbb{E}[|A_T|^2] < \infty$, and that $A_T = L^2 - \lim_{m \to \infty} (A_m)_T$. Hence, A_T is in the second-order Itô chaos; it follows from hypercontractivity (pp. 61-63 of [18], for example), that $\mathbb{E}[|A_T|^j] < \infty$ for all $1 \le j < \infty$. Combining this with the above remark, we find that

$$\|Y_T\|_{1,j}^j = \mathbb{E}[|Y_T|^j] + \mathbb{E}\left[\|DY_T\|_{(\mathcal{H}^2)^*}^j\right]$$
$$\le \mathbb{E}[|B_T|^j] + \mathbb{E}[|A_T|^j] + \mathbb{E}\left[\|DY_T\|_{(\mathcal{H}^2)^*}^j\right] < \infty.$$

$\square$

3.2. $(\det \gamma)^{-1} \in L^{\infty-}(\mathcal{W}^2, \mathbb{P})$. We begin by recording some more general results, which will be useful in proving integrability of $(\det \gamma)^{-j}$.

Lemma 3.10. *Suppose that X is a non-negative random variable such that, for each $j \ge 1$, there exists a constant $C_j > 0$ for which*

$$\mathbb{E}\left[e^{-sX}\right] \le C_j s^{-j} \quad \forall s \ge 1.$$

Then $X^{-1} \in L^{\infty-}$.

Proof. Fix $j \ge 1$. We note that for any $k \ge 0$,

$$\int_0^\infty s^{j-1} e^{-ks} \, ds = k^{-j} \Gamma(j),$$

where Γ denotes the standard Gamma function. By letting $k = X$, we find that

$$\mathbb{E}[X^{-j}] = \frac{1}{\Gamma(j)} \mathbb{E}\left[\int_0^\infty s^{j-1} e^{-sX} \, ds\right] = \frac{1}{\Gamma(j)} \int_0^\infty s^{j-1} \mathbb{E}\left[e^{-sX}\right] \, ds.$$

Using the assumption given, we can see that this quantity is clearly finite. $\square$

Theorem 3.11 (see Melcher [16, pp.26-27]). *Let $(\mathcal{W}, \mathcal{B}, \mathbb{P})$ be a Gaussian measure space with associated Cameron-Martin space $\mathcal{H}$, and suppose $\Phi : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ is a bounded non-negative quadratic form. Then the operator $\hat{\Phi} : \mathcal{H} \to \mathcal{H}$ given by*

$$\Phi(h, k) = \langle \hat{\Phi} h, k \rangle_{\mathcal{H}}$$

is trace-class. In addition, if $\hat{\Phi}$ is not a finite rank operator, then

$$\Phi^{-1} \in L^{\infty-}(\mathcal{W}, \mathbb{P}).$$

Proof. By Theorem 5.3.32 of [22], we have that for a set of independent, identically distributed standard normal random variables $\{\xi_n\}_{n=1}^{\infty}$, the series $B^N := \sum_{n=1}^{N} \xi_n h_n$ converges in $\mathcal{W}$ to B $\mathbb{P}$-a.s. and in all $L^j, j \geq 1$ as $N \to \infty$, and

$$\text{Law}\left(\sum_{n=1}^{\infty} \xi_n h_n\right) = \mathbb{P}.$$

In particular, the fact that $\mathbb{E}\|B^N - B\|_{\mathcal{W}}^2 \to 0$ implies that

$$\mathbb{E}|\Phi(B^N, B^N)| \to \mathbb{E}|\Phi(B, B)|,$$

and Fernique's theorem allows us to conclude that

$$\sum_{n=1}^{\infty} \left\langle \hat{\Phi} h_n, h_n \right\rangle = \lim_{N\to\infty} \sum_{n=1}^{N} \Phi(h_n, h_n)$$
$$= \lim_{N\to\infty} \mathbb{E}\left[\Phi(B^N, B^N)\right]$$
$$= \mathbb{E}[\Phi(B)] \leq C\mathbb{E}[\|B\|_{\mathcal{W}}^2] < \infty.$$

Thus, $\hat{\Phi}$ is trace-class.

Suppose that $\hat{\Phi}$ is not finite rank. Since $\hat{\Phi}$ is compact, there exists an orthonormal basis $\{h_n\}_{n=1}^{\infty} \subset \mathcal{H}$ for which $\hat{\Phi} h_n = \lambda_n h_n$; our assumption guarantees that $\#\{n : \lambda_n > 0\} = \infty$. Using this, it is easy to check that

$$\Phi(B^N, B^N) = \left\langle \hat{\Phi} B^N, B^N \right\rangle_{\mathcal{H}} = \sum_{n=1}^{N} \lambda_n \xi_n^2$$

and so

$$\Phi(B, B) = \lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n \xi_n^2.$$

We will let $K_N := \#\{1 \leq n \leq N : \lambda_n > 0\}$; it is clear that $K_N \overset{N\to\infty}{\longrightarrow} \infty$. Therefore, for each fixed N and positive s,

$$\mathbb{E}\left[\exp\left(-s\Phi\left(B, B\right)\right)\right] = \mathbb{E}\left[\exp\left(-s \lim_{N\to\infty} \sum_{n=1}^{N} \lambda_n \xi_n^2\right)\right]$$
$$\leq \mathbb{E}\left[\exp\left(-s \sum_{n=1}^{N} \lambda_n \xi_n^2\right)\right]$$
$$= \prod_{n=1}^{N} \left(\frac{1}{2\lambda_n s + 1}\right)^{\frac{1}{2}} \leq C_N s^{-\frac{K_N}{2}}.$$

Applying Lemma 3.10 finishes the proof. $\qquad\square$

In order to apply Theorem 3.11, we will explicitly calculate a formula for the determinant of the Malliavin covariance matrix associated to Y.

Lemma 3.12. *Given any $a \neq 0$, $C \in M_{m,n}(\mathbb{R})$, and $D \in M_n(\mathbb{R})$, one has that*
$$\det \begin{bmatrix} aI_m & C \\ C^{tr} & D \end{bmatrix} = a^m \left(\det(D - a^{-1}C^{tr}C) \right),$$
where C^{tr} is the transpose of C.

Proof. This claim follows immediately when one writes
$$\begin{bmatrix} aI_m & C \\ C^{tr} & D \end{bmatrix} = \begin{bmatrix} aI_m & 0 \\ C^{tr} & I_n \end{bmatrix} \begin{bmatrix} I_m & a^{-1}C \\ 0 & D - a^{-1}C^{tr}C \end{bmatrix}.$$
$\square$

Proposition 3.13. *Define the map $\gamma : \mathcal{W}_p^2 \to M^3(\mathbb{R})$ in the following manner:*
$$\gamma(\omega) := \begin{bmatrix} T^{2H}I_2 & (Q\omega)(T) \\ [(Q\omega(T))]^{tr} & \|Q\omega\|_{\mathcal{H}^2}^2 \end{bmatrix}$$
Also, define the quadratic form Φ on $\mathcal{W}_p^2$ as follows:
$$\Phi(\omega) = T^{4H}\|Q\omega\|_{\mathcal{H}^2}^2 - T^{2H}|Q\omega(T)|^2.$$

Then

(1) $DY_T(DY_T)^* = \gamma$ *a.s.*
(2) $\Phi = \det \gamma$.

Proof.   1. We begin by calculating the adjoint operator $(DY)^*$.
Recall that $R_t^i h = h^i(t)e_i$. For a fixed $\lambda = (\lambda^1, \lambda^2) \in \mathbb{R}^2$, $h \in \mathcal{H}^2$, and $i = 1, 2$,
$$\begin{aligned} \left\langle (R_T^i)^*\lambda, h \right\rangle_{\mathcal{H}^2} &= \lambda \cdot h^i(T)e_i \\ &= \lambda^i h^i(T) = \left\langle \lambda^i R(T, \cdot)e_i, h \right\rangle_{\mathcal{H}^2}. \end{aligned}$$

Thus, one has that
$$R_T^i(R_T^j)^*\lambda = R_T^i \lambda^j R(T, \cdot)e_j = \delta_{ij}T^{2H}\lambda^j.$$

Suppose $\mathbf{x} \in \mathbb{R}^3$.
$$\begin{aligned} \left\langle (DY_T)^*\mathbf{x}, k \right\rangle &= \mathbf{x} \cdot DY_T k \\ &= \mathbf{x} \cdot (k(T), \langle QB, k \rangle) \\ &= (x^1, x^2) \cdot k(T) + x^3(\langle QB, k \rangle) \\ &= \left\langle (R_T^1)^*(x^1, x^2) + (R_T^2)^*(x^1, x^2) + x^3 QB, k \right\rangle \\ &= \left\langle x^1 R(T, \cdot)e_1 + x^2 R(T, \cdot)e_2 + x^3 QB, k \right\rangle. \end{aligned}$$

Using this, we may now verify the claim:

$$
\begin{aligned}
DY_T(DY_T)^*\mathbf{x} &= DY_T\left(x^1 R(T,\cdot)e_1 + x^2 R(T,\cdot)e_2 + x^3 QB\right) \\
&= DY_T(x^1 R(T,\cdot)e_1) + DY_T(x^2 R(T,\cdot)e_2) + DY_T(x^3 QB) \\
&= (T^{2H}x^1, 0, \langle QB, x^1 R(T,\cdot)e_1\rangle) \\
&\quad + (0, T^{2H}x^2, \langle QB, x^2 R(T,\cdot)e_2\rangle) \\
&\quad + (x^3 QB^1(T), x^3 QB^2(T), \langle QB, x^3 QB\rangle) \\
&= \begin{bmatrix} T^{2H}I_2 & QB(T) \\ (QB(T))^{tr} & \|QB\|_{\mathcal{H}^2}^2 \end{bmatrix} \mathbf{x}.
\end{aligned}
$$

2. This follows immediately from Lemma 3.12.

$\square$

Lemma 3.14. *The quadratic form Φ is positive semidefinite and has a trivial nullspace.*

Proof. The Cauchy-Schwarz inequality allows us to see that

$$
|(Q\omega)^i(T)|^2 = |\langle R(T,\cdot), (Q\omega)^i\rangle_{\mathcal{H}}|^2 \le \|R(T,\cdot)\|_{\mathcal{H}}^2 \|(Q\omega)^i\|_{\mathcal{H}}^2 = T^{2H}\|(Q\omega)^i\|_{\mathcal{H}}^2.
$$

Hence, Φ is non-negative and $\Phi = 0$ if and only if $Q\omega = \mathbf{c}R(T,\cdot)$ for some constant vector $\mathbf{c} \in \mathbb{R}^2$. By the definition of Q, it would then follow that

$$
0 = \int_0^T \left(\tilde\omega(t) - \tilde\omega(T) + \mathbf{c}\right) R(dt,\cdot)
$$

which implies by Proposition 3.4 that

$$
0 = \int_0^T \left(\tilde\omega(t) - \tilde\omega(T) + \mathbf{c}\right) \cdot dh(t) \quad \left(\forall\, h \in \mathcal{H}^2\right).
$$

As a result of Lemma 31 of [9], one has that $\mathcal{C}_c^\infty(0,T) \subset \mathcal{H}$. Hence, we can conclude that $(\tilde\omega(t) - \tilde\omega(T)) + \mathbf{c}$ is constant on $[0,T]$, which implies that $\tilde\omega(t)$ is constant as well. Thus $\omega(t) = \omega(0) = 0$ for all $t \in [0,T]$. $\square$

Corollary 3.15. $(\Phi)^{-1} \in L^{\infty-}(\mathcal{W}_p^2, \mathbb{P})$.

Proof. Combining Lemma 3.14 and Theorem 3.11 gives us the desired result. $\square$

4. APPENDIX

4.1. **Two-Dimensional r-Variation for Covariance of Fractional Brownian Motion.** Fix $0 < H < 1/2$ and $T > 0$. We will, as

usual, denote by $R : [0, T]^2 \longrightarrow \mathbb{R}$ the covariance function for fractional Brownian motion with Hurst parameter H on $[0, T]$; that is,

$$\mathbb{E}[B_s^H B_t^H] = R(s, t) := \frac{1}{2} \left[s^{2H} + t^{2H} - |t - s|^{2H} \right].$$

To this covariance function, we may associate a finitely addivite signed measure μ_R on the algebra generated by rectangles of the form $\{(a, b] \times (c, d] \subset (0, T]^2\}$ such that

$$\mu_R((a, b] \times (c, d]) = R(b, d) - R(a, d) - R(b, c) + R(a, c)$$

$$= \frac{1}{2} \left(|d - a|^{2H} + |c - b|^{2H} - |d - b|^{2H} - |c - a|^{2H} \right).$$

It is easy to check the covariance of the process increments $B_b^H - B_a^H$ and $B_d^H - B_c^H$ is given by the μ_R-measure of the rectangle $(a, b] \times (c, d]$, or

$$\mathbb{E}[(B_d^H - B_c^H)(B_b^H - B_a^H)] = \mu_R((a, b] \times (c, d]).$$

Lemma 4.1. *Suppose $0 \le a < b < c < d \le T$. Then the process increments $B_d^H - B_c^H$ and $B_b^H - B_a^H$ are negatively correlated; i.e., $\mu_R((a, b] \times (c, d]) < 0$.*

Proof. We begin by noting that the following relations hold:

(1) $(d - a) + (c - b) = (d - b) + (c - a)$.
(2) $(d - a) > [(d - b) \vee (c - a)] := \max[(d - b), (c - a)]$.
(3) $(c - b) < [(d - b) \wedge (c - a)] := \min[(d - b), (c - a)]$.

Let $f(\alpha, \beta) = \alpha^{2H} + \beta^{2H}$. Then for any positive constant C, one can check that on the region $\{(\alpha, \beta) : \alpha \ge 0, \beta \ge 0, \alpha + \beta = C\}$, f has a maximum at $(\frac{C}{2}, \frac{C}{2})$ and decreases as either α or β are increased. Thus, $f(d - a, c - b) < f(d - b, c - a)$, and the claim is proven. $\square$

Throughout the sequel, we will make use of the fact that for $H < \frac{1}{2}$, one has the inequalities

$$(x + y)^{2H} - x^{2H} \le y^{2H}$$

$$x^{\frac{1}{2H}} + y^{\frac{1}{2H}} \le (x + y)^{\frac{1}{2H}}$$

for all $x, y \ge 0$.

Lemma 4.2. *Let μ_R be the measure defined above. Then for any intervals $(a, b], (c, d] \subset [0, T]$, one has the bound*

$$|\mu_R((a, b] \times (c, d])| \le (b - a)^{2H} \wedge (d - c)^{2H}.$$

Proof. We will need to consider three possible cases:

(1) One interval is nested within the other,
(2) the intervals partially overlap, or
(3) the intervals are disjoint.

Case 1. $a \leq c < d \leq b$.

In this scenario, the claimed upper bound is clearly $(d - c)^{2H}$. Using this, we have that

$$
\begin{aligned}
|\mu_R((a, b] \times (c, d])| &= \frac{1}{2} \left| (d - a)^{2H} + (b - c)^{2H} - (b - d)^{2H} - (c - a)^{2H} \right| \\
&\leq \frac{1}{2} \left((d - a)^{2H} - (c - a)^{2H} \right) \\
&\qquad + \frac{1}{2} \left((b - c)^{2H} - (b - d)^{2H} \right) \\
&\leq (d - c)^{2H}.
\end{aligned}
$$

Case 2. $a < c \leq b < d$.

In this case, we know that

$$
\begin{aligned}
|\mu_R((a, b] \times (c, d])| &= \frac{1}{2} \left| (d - a)^{2H} + (b - c)^{2H} - (d - b)^{2H} - (c - a)^{2H} \right| \\
&\leq \frac{1}{2} \left((d - a)^{2H} - (d - b)^{2H} \right) \\
&\qquad + \frac{1}{2} \left| (b - c)^{2H} - (c - a)^{2H} \right| \\
&\leq \frac{1}{2}(b - a)^{2H} + \frac{1}{2} \left((b - c)^{2H} \vee (c - a)^{2H} \right) \\
&\leq (b - a)^{2H}.
\end{aligned}
$$

In a similar manner,

$$
\begin{aligned}
|\mu_R((a, b] \times (c, d])| &= \frac{1}{2} \left| (d - a)^{2H} + (b - c)^{2H} - (d - b)^{2H} - (c - a)^{2H} \right| \\
&\leq \frac{1}{2} \left((d - a)^{2H} - (c - a)^{2H} \right) \\
&\qquad + \frac{1}{2} \left| (b - c)^{2H} - (d - b)^{2H} \right| \\
&\leq \frac{1}{2}(d - c)^{2H} + \frac{1}{2} \left((b - c)^{2H} \vee (d - b)^{2H} \right) \\
&\leq (d - c)^{2H}.
\end{aligned}
$$

Case 3. $a < b \leq c < d$.

Here, we will use the concavity inequality twice to generate the desired bound. Firstly, we calculate that

$$
\begin{aligned}
|\mu_R((a,b] \times (c,d])| &= \frac{1}{2} \left| (d-a)^{2H} + (c-b)^{2H} - (d-b)^{2H} - (c-a)^{2H} \right| \\
&\leq \frac{1}{2} \left((d-a)^{2H} - (c-a)^{2H} \right) \\
&\qquad + \frac{1}{2} \left((d-b)^{2H} - (c-b)^{2H} \right) \\
&\leq (d-c)^{2H}.
\end{aligned}
$$

In much the same manner, we find that

$$
\begin{aligned}
|\mu_R((a,b] \times (c,d])| &= \frac{1}{2} \left| (d-a)^{2H} + (c-b)^{2H} - (d-b)^{2H} - (c-a)^{2H} \right| \\
&\leq \frac{1}{2} \left((d-a)^{2H} - (d-b)^{2H} \right) \\
&\qquad + \frac{1}{2} \left((c-a)^{2H} - (c-b)^{2H} \right) \\
&\leq (b-a)^{2H}.
\end{aligned}
$$

$\square$

Theorem 4.3. *Let $r := \frac{1}{2H} > 1$. Then the function R has finite two-dimensional r-variation over $[0,T]^2$; more specifically,*

$$
\|R\|_r^{(2D)} \leq (5T)^{2H}.
$$

Proof. Let

$$
\begin{aligned}
\Pi &:= \{ s_0 := 0 < s_1 < \ldots < s_M := T \}, \\
\Psi &:= \{ t_0 := 0 < t_1 < \ldots < t_N := T \}
\end{aligned}
$$

be two partitions of $[0,T]$. Fix a $j \in \{1, \ldots, T\}$. We will let A be the unique integer such that $s_{A-1} \leq t_{j-1} < s_A$, and $L \geq A$ will denote the unique integer for which $s_{L-1} < t_j \leq s_L$.
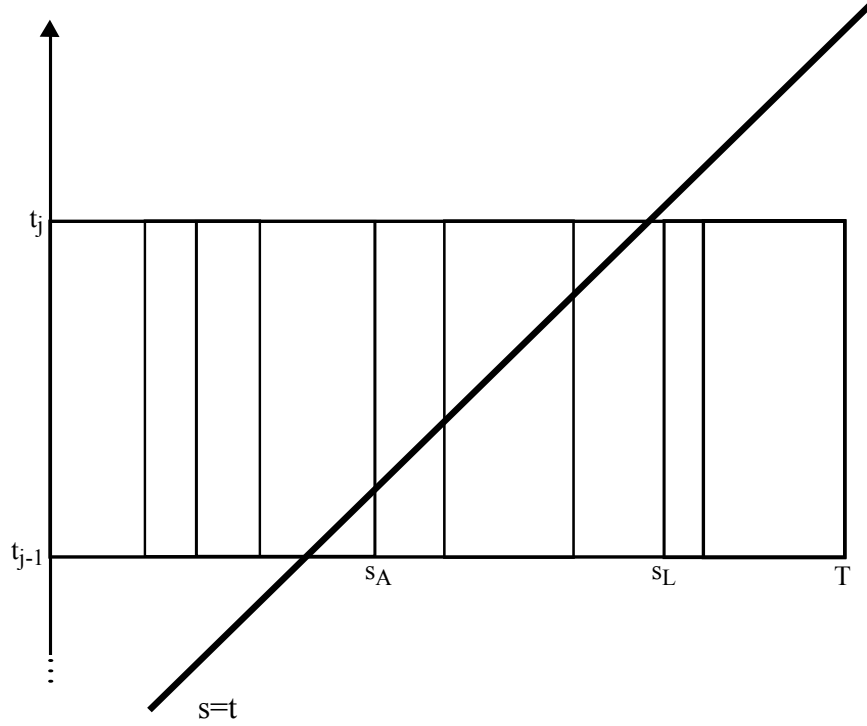
FIGURE 1. An example partition in s for a fixed strip $t_{j-1} < t \leq t_j$.

As usual, we define

$$\Delta_{ij} R := \mu_R\left((s_{i-1}, s_i] \times (t_{j-1}, t_j]\right).$$

Then

$$(4.1) \quad \sum_{i=1}^{M} |\Delta_{ij} R|^r \leq \sum_{i=1}^{A-1} |\Delta_{ij} R|^r + |\Delta_{Aj} R|^r$$

$$+ \sum_{i=A+1}^{L-1} |\Delta_{ij} R|^r + |\Delta_{Lj} R|^r + \sum_{i=L+1}^{M} |\Delta_{ij} R|^r.$$

It follows from Lemma 4.2 that

$$(4.2) \qquad |\Delta_{Aj} R|^r \leq (t_j - t_{j-1}), \qquad |\Delta_{Lj} R|^r \leq (t_j - t_{j-1}).$$

Lemma 4.2 also implies that $|\Delta_{ij} R|^r \leq (s_i - s_{i-1})$; hence, we may use telescoping to bound the third term:

$$(4.3) \quad \sum_{i=A+1}^{L-1} |\Delta_{ij} R|^r \leq \sum_{i=A+1}^{L-1} (s_i - s_{i-1}) = (s_{L-1} - s_A) \leq (t_j - t_{j-1}).$$

Let us now focus on the first and last terms of Equation (4.1). Note that on each of these sums, Lemma 4.1 implies that $\Delta_{ij} R < 0$. We may use this fact along with Lemma 4.2 to see that

$$
\sum_{i=1}^{A-1} |\Delta_{ij} R|^r + \sum_{i=L+1}^{M} |\Delta_{ij} R|^r
$$

$$
\leq \left(\sum_{i=1}^{A-1} |\Delta_{ij} R| \right)^r + \left(\sum_{i=L+1}^{M} |\Delta_{ij} R| \right)^r
$$

$$
= \left| \sum_{i=1}^{A-1} \Delta_{ij} R \right|^r + \left| \sum_{i=L+1}^{M} \Delta_{ij} R \right|^r
$$

$$
= |\mu_R ((0, s_{A-1}] \times (t_{j-1}, t_j])|^r
$$

$$
+ |\mu_R ((s_L, T] \times (t_{j-1}, t_j])|^r
$$

(4.4)
$$
\leq 2(t_j - t_{j-1}).
$$

Combining Equations (4.1)–(4.4) allows us to conclude that

$$
\sum_{i=1}^{M} |\Delta_{ij} R|^r \leq 5(t_j - t_{j-1}).
$$

Hence,

$$
\sum_{j=1}^{N} \sum_{i=1}^{M} |\Delta_{ij} R|^r \leq \sum_{j=1}^{N} 5(t_j - t_{j-1}) = 5T.
$$

This completes the proof, since the two-dimensional r-variation of R is given as

$$
\|R\|_r^{(2D)} = \left(\sup_{\Pi, \Psi \in \mathcal{P}[0,T]} \sum_{\Pi} \sum_{\Psi} |\Delta_{ij} R|^r \right)^{\frac{1}{r}} \leq (5T)^{2H}.
$$

$\square$

4.2. **Restriction of Gaussian Measures.** At first blush, it may seem natural to have our process B have the classical Wiener space $\mathcal{W}^2 := \mathcal{C}([0, T], \mathbb{R}^2)$ as its sample space. However, doing so is not ideal, since many of the operators we will be considering are only defined on smaller spaces, such as the p-variation spaces.

We begin with a general result regarding σ-algebras.

Lemma 4.4. *Let X be any real separable Banach space and $\mathcal{L}$ be any Then $\|\cdot\|_X$ is $\sigma(\mathcal{L})$ – measurable if and only if $\mathcal{B}_X = \sigma(\mathcal{L})$.*

Proof. It it easy to see that, in any case, $\sigma(\mathcal{L}) \subset \mathcal{B}_X$. Also, since $\|\cdot\|_X$ is continuous it is always Borel measurable; therefore, if $\mathcal{B}_X = \sigma(\mathcal{L})$ then $\|\cdot\|_X$ is clearly $\sigma(\mathcal{L})$ – measurable.

Suppose that $\|\cdot\|_X$ is $\sigma(\mathcal{L})$ – measurable; then for each $x_0 \in \sigma(\mathcal{L})$, $\|\cdot - x_0\|_X$ is also $\sigma(\mathcal{L})$ – measurable, and $x \to x - x_0$ is $\sigma(\mathcal{L})/\sigma(\mathcal{L})$ – measurable. From this observation, it follows that $\sigma(\mathcal{L})$ contains all balls in X. Since X is separable, every open subset of X may be written as a countable union of open balls. It follows, then, that $\sigma(\mathcal{L})$ contains all open subsets of X and therefore that $\mathcal{B}_X \subset \sigma(\mathcal{L})$. $\qquad\qquad\square$

Theorem 4.5. *Suppose $(X, \mathcal{B} = \mathcal{B}_X, \mu)$ is a Gaussian probability space, and $\tilde{X}$ is a linear subspace of X. Also let $\| \cdot \|_{\tilde{X}}$ is a norm on $\tilde{X}$ such that*

(1) *The space $(\tilde{X}, \| \cdot \|_{\tilde{X}})$ is a separable Banach space,*
(2) *The embedding of $\tilde{X}$ into X is continuous,*
(3) *$\tilde{X} \in \mathcal{B}$ and $\mu(\tilde{X}) = 1$,*
(4) *$\tilde{\mathcal{B}} := \mathcal{B}_{\tilde{X}} = \{A \cap \tilde{X} : A \in \mathcal{B}\}$.*

Then $\tilde{\mu} := \mu|_{\tilde{X}}$ is a Gaussian measure and $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ is a Gaussian probability space. Furthermore, (X, μ) and $(\tilde{X}, \tilde{\mu})$ share the same Cameron-Martin space $\mathcal{H}$.

Proof. Let $R_{\pi/4} : X \times X \to X \times X$ is the rotation map defined by

$$R_{\pi/4}(x, y) = \left(\frac{\sqrt{2}}{2}(x - y), \frac{\sqrt{2}}{2}(x + y) \right);$$

then by the rotational invariance of Gaussian measures (see, for example, Theorem 3.1.1 of [4]), proving the statement that $\tilde{\mu}$ is Gaussian is equivalent to proving that

$$\int_{\tilde{X} \times \tilde{X}} f(x, y) \, d\tilde{\mu}(x) d\tilde{\mu}(y) = \int_{\tilde{X} \times \tilde{X}} f \circ R_{\pi/4}(x, y) \, d\tilde{\mu}(x) d\tilde{\mu}(y)$$

for any bounded $\tilde{\mathcal{B}} \times \tilde{\mathcal{B}}$-measurable function f. Let f be such a function; since $\tilde{X}$ is of full μ measure, we may extend f to an $\mathcal{B} \times \mathcal{B}$-measurable function (which we shall also refer to as f) such that $\int_{\tilde{X} \times \tilde{X}} f d\mu d\mu = \int_{X \times X} f d\mu d\mu$ (this extension may be done by setting a function equal to f on $\tilde{X} \times \tilde{X}$ and equal to zero on the complement, for example).

Then it follows that

$$\int_{\tilde{X}\times\tilde{X}} f(x,y)\,d\tilde{\mu}(x)d\tilde{\mu}(y) = \int_{\tilde{X}\times\tilde{X}} f(x,y)\,d\mu(x)d\mu(y)$$

$$= \int_{X\times X} f(x,y)\,d\mu(x)d\mu(y)$$

$$= \int_{X\times X} f\circ R_{\pi/4}(x,y)\,d\mu(x)d\mu(y)$$

$$= \int_{\tilde{X}\times\tilde{X}} f\circ R_{\pi/4}(x,y)\,d\mu(x)d\mu(y)$$

$$= \int_{\tilde{X}\times\tilde{X}} f\circ R_{\pi/4}(x,y)\,d\tilde{\mu}(x)d\tilde{\mu}(y).$$

This proves the first assertion.

To see the equivalence of Cameron-Martin spaces, we recall that $J : L^2(X,\mu) \to X$, defined by

$$Jf := \int_X xf(x)\,d\mu(x),$$

maps onto $\mathcal{H}$. Again, by virtue of μ being fully supported on $\tilde{X}$, we may extend any element of $L^2(\tilde{X},\tilde{\mu})$ to an element of $L^2(X,\mu)$; thus it is easy to see that $J(L^2(\tilde{X},\tilde{\mu})) = J(L^2(X,\mu)) = \mathcal{H}$, as desired. □

Remark 4.6. *An alternate proof of the equivalence of Cameron-Martin spaces may be found in Proposition 2.8 of* [8].

Let us now focus on restricting the law of fractional Brownian motion with Hurst parameter $1/3 < H < 1/2$ to a variational space. The standard Gaussian space on which fBm is realized is $(\mathcal{W}, \mathcal{B}, \mathbb{P})$, where $\mathcal{W} = \{\omega \in \mathcal{C}([0,T],\mathbb{R}) : \omega(0) = 0\}$ and $\mathbb{P} = \mathrm{Law}(B^H)$. Pick $0 < \epsilon << 1$ and fix $p := 1/H + \epsilon$. Let $\phi_t, 0 \leq t \leq T$ denote the evaluation map on $\mathcal{W}$; i.e., $\phi_t(x) = x(t)$ for any $x \in \mathcal{W}$. Since

$$\| \cdot \|_{\mathcal{W}} = \sup_{0\leq t\leq T} \phi_t,$$

it follows that $\| \cdot \|_{\mathcal{W}}$ is a $\sigma(\{\phi_t : 0 \leq t \leq T\})$-measurable function, and by Lemma 4.4, it then follows that $\sigma(\{\phi_t : 0 \leq t \leq T\}) = \mathcal{B}_{\mathcal{W}}$. Recall that we have defined the p-variation norm on $\mathcal{W}$ by

$$\|x\|_p = \sup_{\Pi\in\mathcal{P}[0,T]} \left(\sum_{i=1}^{(\#\Pi)} |\Delta_i x|^p \right)^{\frac{1}{p}}.$$

Recall that we have defined the space

$$\mathcal{W}_p = \overline{\{x \in \mathcal{C}_\infty([0, T], \mathbb{R}) : x(0) = 0\}}^{\|\cdot\|_p}.$$

By Corollary 5.35 and Proposition 5.38 of [9], this space is a separable Banach space under the p-variation norm and contains all q-variation paths starting at zero for any $1 \le q < p$. Note that for $x \in \mathcal{W}_p$, Hölder's inequality implies that for any $t \in [0, T]$,

$$\begin{aligned}
|x(t)| &= |x(t) - x(0)| \\
&\le |x(t) - x(0)| + |x(T) - x(0)| \\
&\le 2^{\frac{p-1}{p}} \left(|x(t) - x(0)|^p + |x(T) - x(0)|^p \right)^{\frac{1}{p}} \\
&\le 2^{\frac{p-1}{p}} \|x\|_p,
\end{aligned}$$

from which it follows that $\|x\|_{\mathcal{W}} \le \|x\|_p$, and so the embedding of $\mathcal{W}_0^p$ into $\mathcal{W}$ is continuous. Observe that we may rewrite the p-variation norm as

$$\| \cdot \|_p = \sup_{\Pi \in \mathcal{P}[0,T]} \left(\sum_{i=1}^{(\#\Pi)} |\phi_{t_i} - \phi_{t_{i-1}}|^p \right)^{\frac{1}{p}}.$$

Thus, $\| \cdot \|_p$ is $\sigma(\{\phi_t|_{\mathcal{W}_p} : 0 \le t \le T\})$-measurable, which implies that $\sigma(\mathcal{L}) = \mathcal{B}_{\mathcal{W}_p}$. Furthermore, by Theorem 5.33 of [9], we know that the space $\mathcal{W}_p$ is equivalent to

$$\left\{ x \in \mathcal{W}_p : \lim_{\delta \to 0} \sup_{\Pi \in \mathcal{P}[0,T] : |\Pi| < \delta} \sum_{i=1}^{\#(\Pi)} |x(t_i) - x(t_{i-1})|^p = 0 \right\}$$

If we now define

$$\alpha_p(x) := \lim_{n \to \infty} \sup_{\Pi \in \mathcal{P}[0,T] \cap \mathbb{N} : |\Pi| < \frac{1}{n}} \sum_{i=1}^{\#(\Pi)} |x(t_i) - x(t_{i-1})|^p,$$

then it follows that α_p is a $\sigma(\{\phi_t|_{\mathcal{W}_p} : 0 \le t \le T\})$-measurable function, and that

$$\mathcal{W}_p = \mathcal{W}_p \cap \{\alpha_p = 0\} \in \mathcal{B}_{\mathcal{W}}.$$

Additionally, we may now use Lemma 4.4 to conclude that

$$\begin{aligned}
\mathcal{B}_{\mathcal{W}_p} &= \sigma(\{\phi_t|_{\mathcal{W}_p} : 0 \le t \le T\}) \\
&= \{A \cap \mathcal{W}_p : A \in \sigma(\{\phi_t|_{\mathcal{W}} : 0 \le t \le T\})\} \\
&= \{A \cap \mathcal{W}_p : A \in \mathcal{B}_{\mathcal{W}}\}.
\end{aligned}$$

Finally, we note that since the paths $t \mapsto B_t^H$ are a.s. Hölder continuous of order $\beta := H\left(1 + \frac{\epsilon H}{2}\right)^{-1} < H$, each such path has finite q-variation for $q = \frac{1}{\beta} = \frac{1}{H} + \frac{\epsilon}{2}$. So by Corollary 5.35 of [9], $\mathbb{P}(\mathcal{W}_p) \ge \mathbb{P}(\mathcal{W}_q) = 1$. Thus, we may appeal to Theorem 4.5 to conclude that $(\mathcal{W}_p, \mathcal{B}_{\mathcal{W}_p}, \mathbb{P}|_{\mathcal{W}_p})$ is also a Gaussian probability space, and that the associated Cameron-Martin space $\mathcal{H}$ coincides with the usual Cameron-Martin space corresponding to $\mathbb{P}$ on $\mathcal{W}$.

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